Coronal loop seismology using the Markov Chain Monte-Carlo techniques

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Outline

1. Bayesian approach
   - Uncertainties estimation
   - Model comparison
   - Posterior predictive check and forecasting
2. Markov Chain Monte Carlo
3. Application to the coronal loop seismology
4. Summary

The Bayes theorem

Improving knowledge

Knowledge before measurements [Prior PDF - $P(\theta)$] $\bigotimes$
Measurements [The Likelihood - $P(D|\theta)$] $\Rightarrow$
Improved knowledge [Posterior PDF - $P(\theta|D)$]

The Bayes theorem

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The Bayes Theorem

$$P(\theta|D) = \frac{P(D|\theta)P(\theta)}{P(D)}$$

The Bayes theorem

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The Bayes Theorem

$$P(\theta|D) = \frac{P(D|\theta)P(\theta)}{P(D)}$$

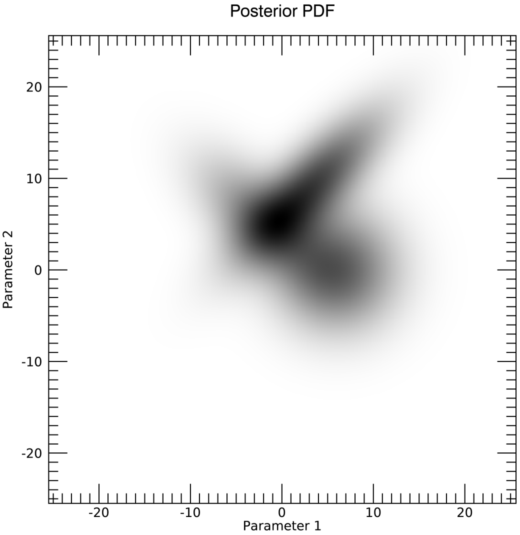
Evidence

$$P(D) = \int P(D|\theta)P(\theta)d\theta$$

Evidence is the quantitative measure of how good the model M describes observational data D.

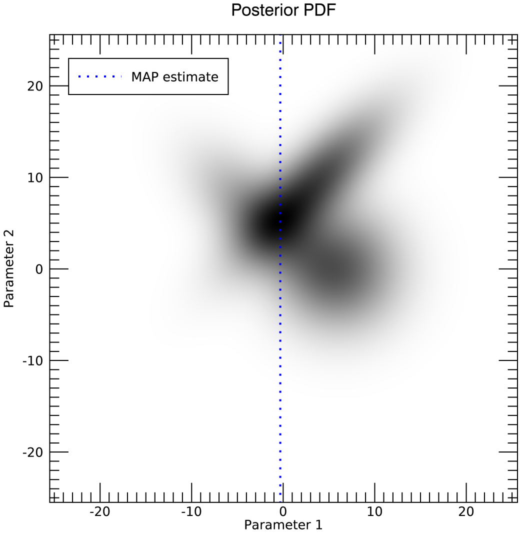
Extracting information from the Posterior PDF

Posterior PDF



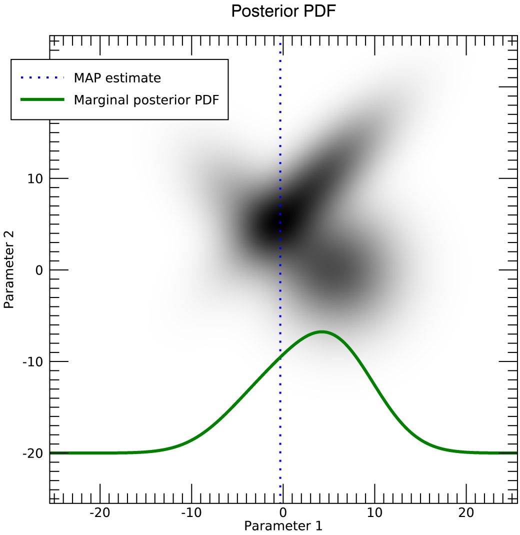
Extracting information from the Posterior PDF

Maximum A Posteriori (MAP) estimate



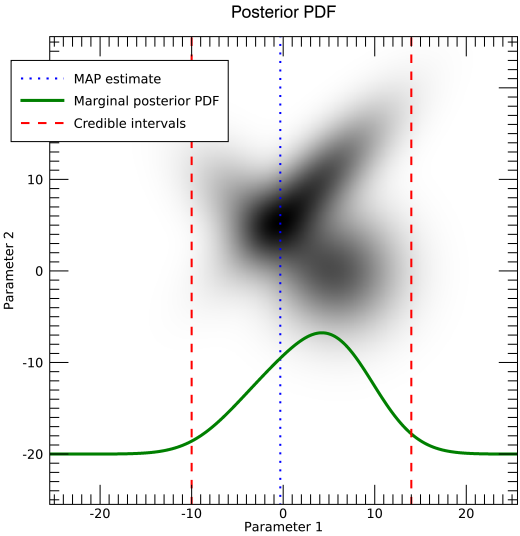
Extracting information from the Posterior PDF

Marginal Posterior PDF



Extracting information from the Posterior PDF

Credible intervals



Model comparisons

Bayes factor

Evidence

$$P(D) = \int P(D|\theta)P(\theta)d\theta$$

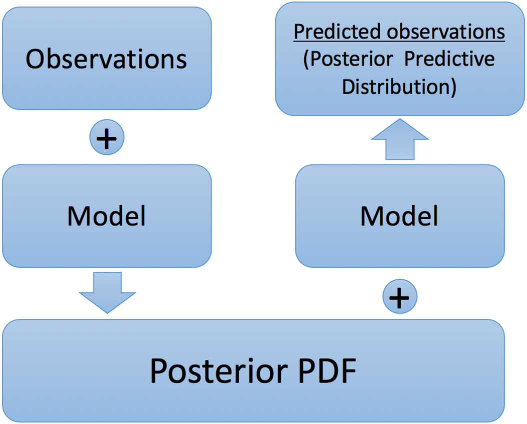
Evidence is the quantitative measure of how good the model M describes observational data D.

$$B_{12} = \frac{\int P(D|\theta, M_1)P(\theta|M_1)d\theta}{\int P(D|\theta, M_2)P(\theta|M_2)d\theta}$$

B_{12}	$2 \ln B_{12}$	Model 1 evidence
1 - 3	0-2	not worth more than a bare mention
3 - 20	2-6	Positive
20 - 150	6-10	Strong
>150	>10	Very strong

Posterior Predictive Distribution

Predictive Check and Forecasting



The curse of dimensions

The determination of the credible intervals requires the computation of the marginal PDFs:

$$P(\theta_i|D) = \int P(\theta_1, \theta_2, \cdots, \theta_N|D) d\theta_{k \neq i}$$

The curse of dimensions

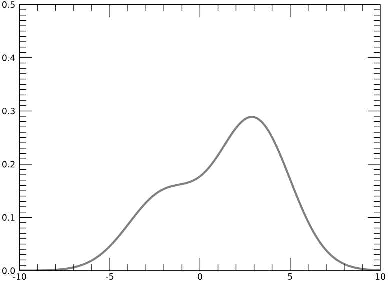
- Analytical integration is often impossible;
- Direct numerical integration every additional parameter increases the computation time by several order of magnitudes.

Solution

- Sample the posterior distribution;
- Approximate marginal PDFs by histograms.

Markov Chain Monte-Carlo

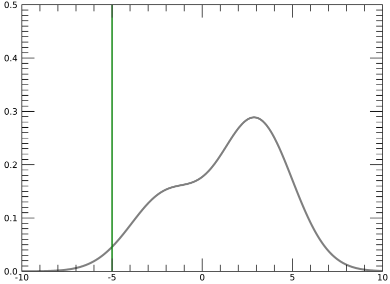
How does MCMC sampler work?



Markov Chain Monte-Carlo

How does MCMC sampler work?

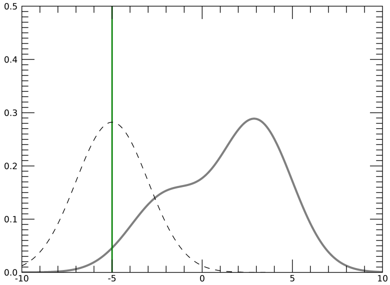
1. Set a walker in the parameter space



Markov Chain Monte-Carlo

How does MCMC sampler work?

1. Set a walker in the parameter space
2. Simulate a new position from the **proposal distribution**



Markov Chain Monte-Carlo

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1. Set a walker in the parameter space
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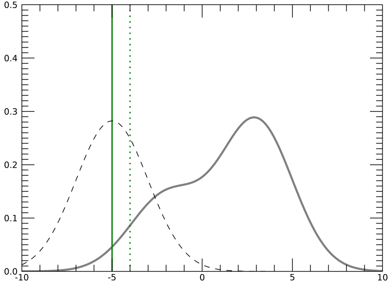
3. Accept or reject the step according a **special rule**



Markov Chain Monte-Carlo

How does MCMC sampler work?

1. Set a walker in the parameter space
2. Simulate a new position from the **proposal distribution**
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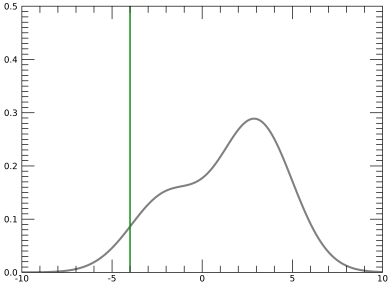
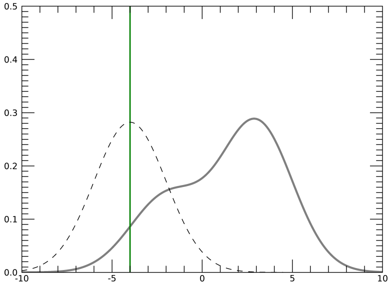
4. Return the current position as a sample from **the target distribution**



Markov Chain Monte-Carlo

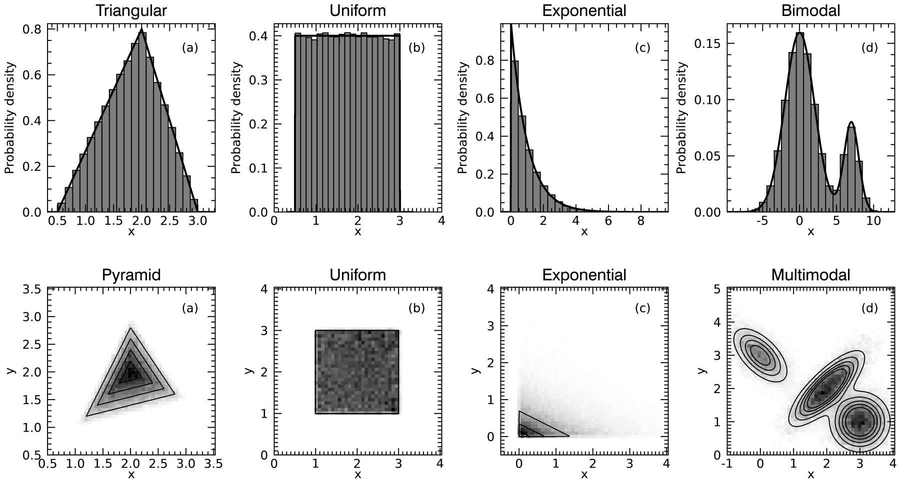
How does MCMC sampler work?

1. Set a walker in the parameter space
2. Simulate a new position from the **proposal distribution**
3. Accept or reject the step according a **special rule**
4. Return the current position as a sample from **the target distribution**
5. Go to **2**



MCMC

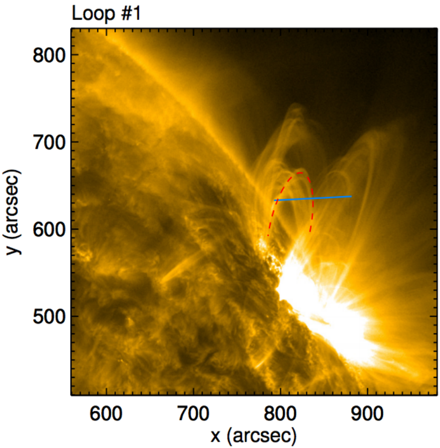
Tests

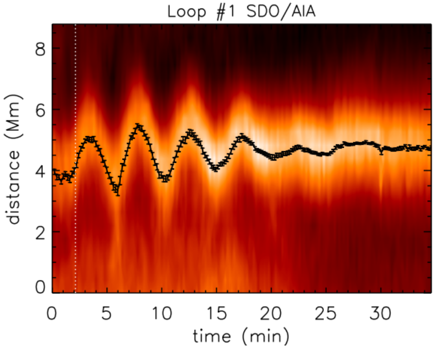


Triangular
Uniform
Exponential
Bimodal
Probability density
x
(a)
(b)
(c)
(d)
Pyramid
Uniform
Exponential
Multimodal
y
x

Observations

7 January 2013 06:38:11 UT

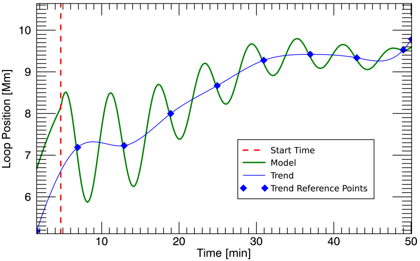




Models of decaying kink oscillations

Parameters:

- A_0 – amplitude
- P – period
- τ – decay time
- t_0 – starting time
- x_0 – initial displacement



$$y(t) = \mathrm{Trend}(t) + \begin{cases} A_0 e^{-\left(\frac{t-t_0}{\tau}\right)^n} \sin\left[\frac{2\pi(t-t_0)}{P} + \arcsin\left(\frac{x_0}{A_0}\right)\right], & t \geq t_0 \\ x_0, & t < t_0 \end{cases}$$

Models

- Exponential decay model (M_{e}): $n = 1$
- Gaussian decay model (M_{G}): $n = 2$

The model

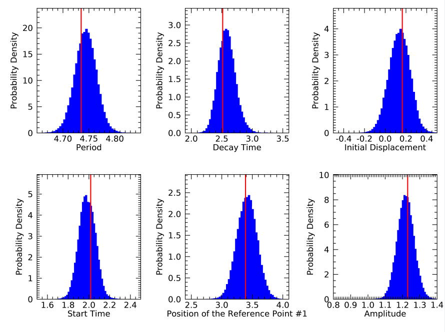
- Observables $D = [Y_1, Y_2, ..., Y_{N_d}]$
- Unknown model parameters $\theta = [A_0, P, \tau, ...]$
- The model $Y_i^m(\theta) = y(\theta) + N_i(0, \sigma^2), 1 < i < N_d$,
  - N_d — number of data points
  - $N(0, \sigma)$ — white noise with zero mean and dispersion of σ^2

The likelyhood

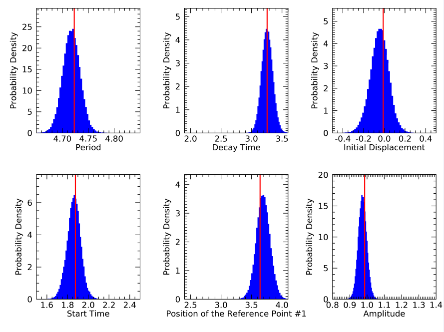
$$P(D|\theta) = \frac{1}{(2\pi\sigma^2)^{\frac{N_d}{2}}} \prod_{i=1}^{N_d} \exp\left\{-\frac{[Y_i - Y_i^m(\theta)]^2}{2\sigma^2}\right\}$$

Parameters Inference

Exponential decay



Gaussian decay



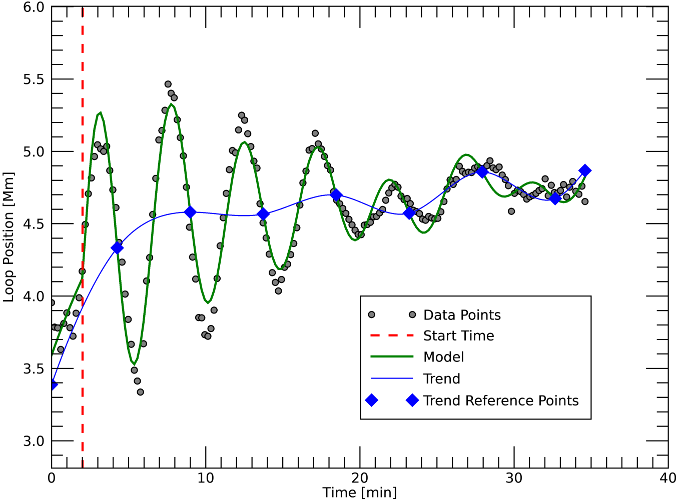
$P(D|M_{\mathrm{e}}) = 1.5 \times 10^{28}$

$P(D|M_{\mathrm{G}}) = 1.0 \times 10^{47}$

$2 \ln B_{\mathrm{e,G}} = 86.7$ (Very Strong evidence for Gaussian decay)

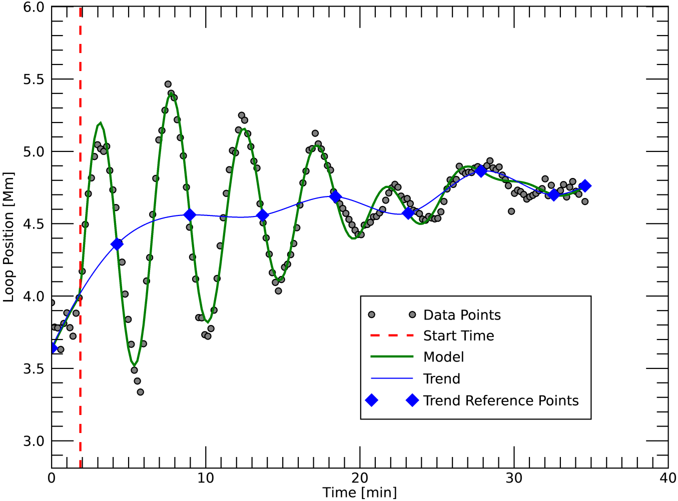
Fitting results

Exponential decay



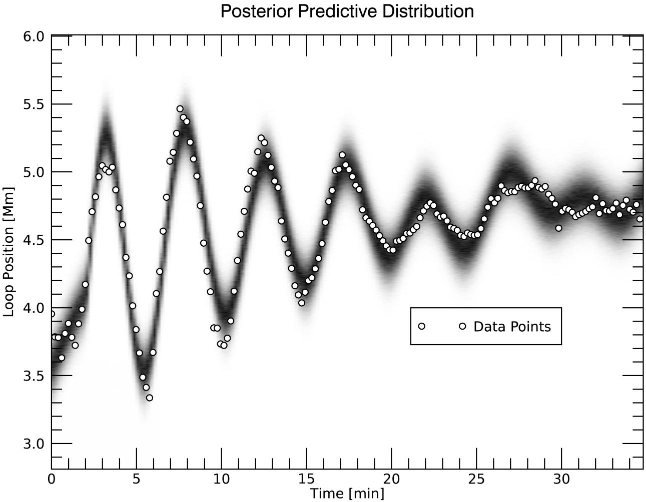
Fitting results

Gaussian decay



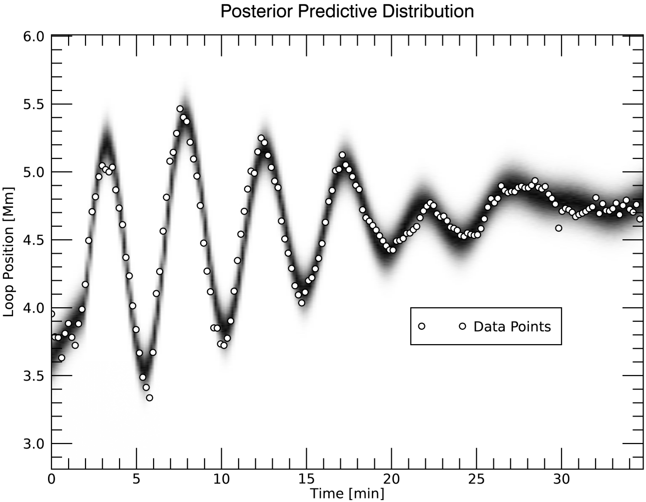
Posterior Predictive Distribution

Exponential decay



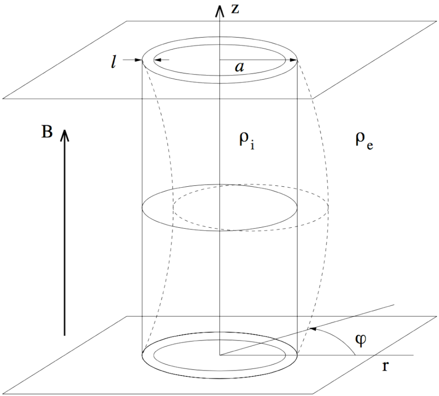
Posterior Predictive Distribution

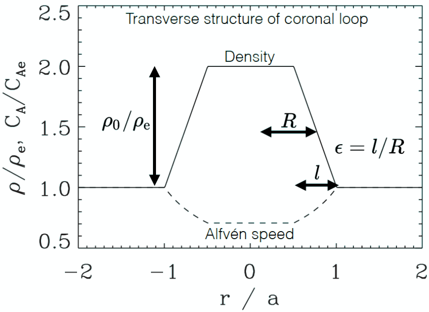
Gaussian decay



Straight cylinder model of a coronal loop

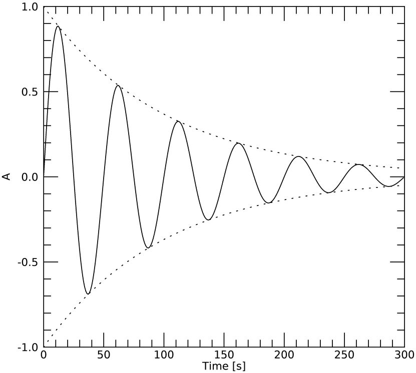
with the transition layer





Exponential decaying profile[1]

$$D(t) = \exp \frac{t}{\tau}, \tau = \frac{2P}{\pi \epsilon \kappa}$$



Unknown parameters

1. $\epsilon = l/R$
2. $\kappa = \frac{\rho_0 - \rho_e}{\rho_0 + \rho_e} = \frac{\rho_0/\rho_e - 1}{\rho_0/\rho_e + 1}$

[1] [Goossens et al., 2002, Ruderman and Roberts, 2002]

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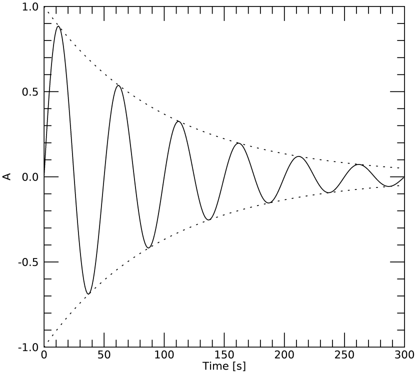

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Observables

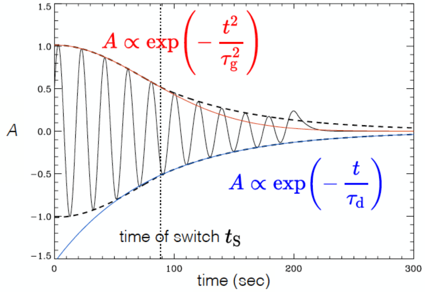
1. Decay time τ

[1] [Goossens et al., 2002, Ruderman and Roberts, 2002]

General decaying profile [2]

$$D(t) = \begin{cases} \exp\left(-\frac{t^2}{2\tau_g^2}\right), & t \le t_s \\ A_s \exp\left(-\frac{t-t_s}{\tau_d}\right), & t > t_s \end{cases}$$

$\tau_g = \frac{2P}{\pi\kappa\epsilon^{1/2}}$, $\tau_d = \frac{4P}{\pi^2\kappa\epsilon}$, $\tau_s = \tau_g^2/\tau_d$



[2] [Hood et al., 2013, Pascoe et al., 2013]

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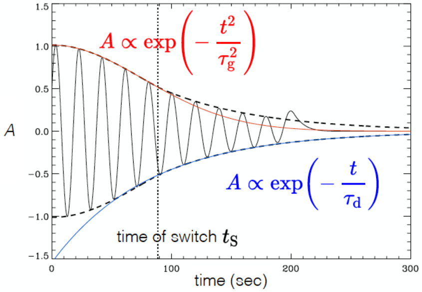

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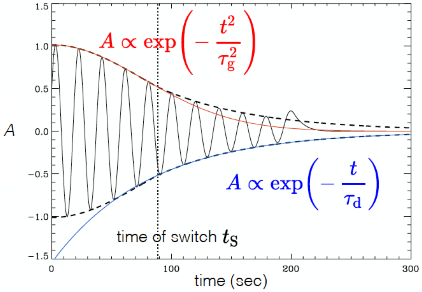

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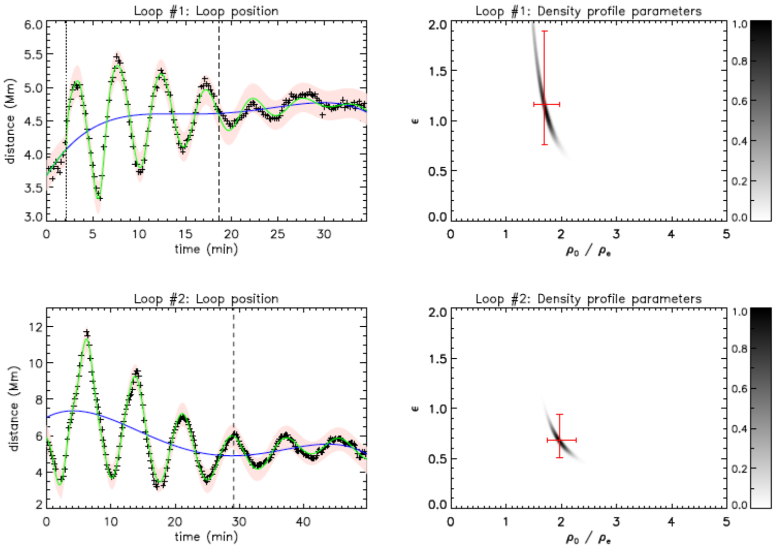
Observables

1. Gaussian decay time τ_g
2. Exponential decay time τ_d

[2][Hood et al., 2013, Pascoe et al., 2013]

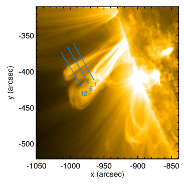
Full Inversion[3]



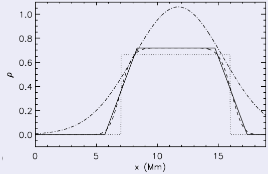
[3][Pascoe et al., 2017a]

Density profile inversion from the EUV intensity [4]

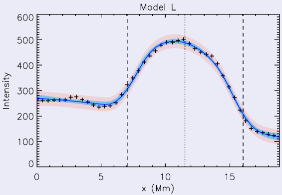
SDO/AIA 171 Å

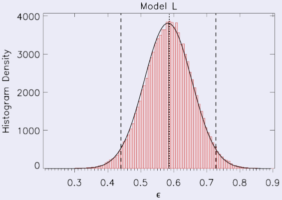


Inferred density profiles



MCMC inference results





[4] [Pascoe et al., 2017b]

Summary

MCMC is a universal method for solving inverse problems.

- Output: probability distribution function for the model parameters
  - Most credible values
  - Robust error estimations
- Comparison of the competing models using the Bayes factor
  - Transparent accounting for the number of parameters
  - Comparison of rather the models than best fits.
- In combination with advanced physical model, it allows us to extract as much reliable information as possible from the observational data
- We inferred transverse density structure of several coronal loops
  - Seismological inversion from kink oscillations
  - Inversion from EUV intensity profiles

Thank you for your attention!

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